Show that:

Th. If function f is continous at $x=a$, then $|f(x)|$ is also continous at $x=a$, but converse is not true.

Proof:- Given that f is continous at $x=a$.

$\Rightarrow$ For a given $\epsilon>0$, $\exists\ \delta>0$ s.t.

$|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$

$\Rightarrow \big|\,|f(x)|-|f(a)|\,\big| \le |f(x)-f(a)| < \epsilon$ with $|x-a|<\delta$

$\because \big|\,|x_1|-|x_2|\,\big| \le |x_1-x_2|$

$\Rightarrow \big|\,|f(x)|-|f(a)|\,\big| < \epsilon$ with $|x-a|<\delta$

$\Rightarrow |f(x)|$ is continous at $x=a$

For converse: ~~converse~~

Let us take example of the function:

$$f(x)=\begin{cases}-1 & ; \ x<a \\ 1 & ; \ x\geq a\end{cases}$$

$\lim_{x\to a}|f(x)| = 1 = f(a)$ This shows that

$|f(x)|$ is continous at $x=a$

But $\lim_{x\to a^-} f(x) = -1$

$\lim_{x\to a^+} f(x) = 1$

$f(a) = 1$

$\Rightarrow$ f(x) is not continous at $x=a$.

H. P.

Q1. Discuss the continuity of the function
$f(x) = |x| + |x-1|$ at $x=0$ and 1.

Sol^n. The given function $f(x)$ can be written as

$$f(x) = \begin{cases} 1-2x, & \text{if } x \le 0 \\ 1, & \text{if } 0 < x < 1 \\ 2x-1, & \text{if } x \geq 1 \end{cases}$$

At $x=0$

R.H.L at $x=0$ $= f(0^+) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} [1] = 1$

LHL at $x=0$ $= f(0^-) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} [1-2(0-h)] = 1$

V.o.f. $= f(0) = 1 - 2.0 = 1$

Here LHL = RHL = V.o.f^n = 1.

At $x=1$

RHL at $x=1$ = $f(1^+) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} [2(1+h)-1]$
$= 1$

LHL at $x=1$ = $f(1^-) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} [1] = 1$

Vof^n at $x=1 = 2(1)-1 = 1$

Here LHL = RHL = vof^n = 1

$\Rightarrow$ f is continuous at $x=1$.

Q2 Test continuity at $x=0$:

$$f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{-1/x}}, & x \neq 0 \\ 0 & x=0 \end{cases}$$

Here LHL $= f(0^-) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\left[\frac{e^{\frac{1}{0-h}}}{1+e^{\frac{1}{0-h}}}\right] = \lim_{h\to 0}\left[\frac{e^{-1/h}}{1+e^{-1/h}}\right]$$

$$\lim_{h\to 0}\left[\frac{1}{e^{1/h}+1}\right] = \frac{1}{\infty} = 0 \quad \text{———— (i)}$$

$$\text{RHL} = f(0^+) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\left[\frac{e^{\frac{1}{0+h}}}{1+e^{\frac{1}{0+h}}}\right]$$

$$= \lim_{h\to 0}\left[\frac{e^{1/h}}{1+e^{1/h}}\right] = \lim_{h\to 0}\left(\frac{1}{e^{-1/h}+1}\right)$$

$$= \frac{1}{0+1} = 1 \quad \text{———— (ii)}$$

V.O.f^n at $x=0$ $= f(0) = 0$ ———— (iii)

Here LHL $\neq$ RHL $\neq$ VOf^n.

$\Rightarrow$ f^n. is ~~cont~~ discontinous at $x=0$.

Q. Test the continuity of the function

$$f(x) = \begin{cases} \frac{x-|x|}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Sol^n Given f^n. is $f(x) = \begin{cases} 2 & ; x<0 \\ 0 & ; x>0 \\ 1 & x=0 \end{cases}$

LHL at $x=0$ = 2
RHL at $x=0$ = 0
VOf^n. at $x=0$ = 1

$\Rightarrow$ f^n is discontinous at $x=0$.